Portfolio 1 Sample

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1. Evaluate $\int_1^5 x^2 e^{2x}\,dx$.

Solution: In order to compute the definite integral $\int_1^5 x^2 e^{2x}\,dx$, we will first compute the indefinite integral $\int x^2 e^{2x}\,dx$ and then evaluate that function at the endpoints.

Part 1: To compute $\int x^2 e^{2x}\,dx$, we will use integration by parts by choosing functions $f_1(x)$ and $g_1'(x)$ to use the equation

$$(*) \int f_1(x)\, g_1'(x)\,dx = f_1(x)\,g_1(x) - \int f_1'(x)\,g_1(x)\,dx.$$

Choose $f_1(x) = x^2$ and $g_1'(x) = e^{2x}$. Differentiating $f_1(x)$ gives $f_1'(x) = 2x$ and integrating $g_1'(x)$ gives $g_1(x) = \frac{1}{2}e^{2x}$.

Placing these four functions into the equation (*), we obtain

$$\int (x^2)(e^{2x})\,dx = (x^2)\left(\tfrac{1}{2}e^{2x}\right) - \int (2x)\left(\tfrac{1}{2}e^{2x}\right)dx.$$

Simplifying, we have

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx.$$

The right-hand side of the equation still has an integral that we need to compute. We will use integration by parts again. Choose $f_2(x) = x$ and $g_2'(x) = e^{2x}$. Differentiating $f_2(x)$ gives $f_2'(x) = 1$ and integrating $g_2'(x)$ gives $g_2(x) = \frac{1}{2}e^{2x}$. Placing these functions into their corresponding spots in the equation (*), we obtain

$$\int xe^{2x}\,dx = (x)\left(\tfrac{1}{2}e^{2x}\right) - \int (1)\left(\tfrac{1}{2}e^{2x}\right)dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx$$

We can evaluate the integral on the right-hand side to obtain

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

(We've added the "$+C$" at this stage since antiderivatives are only unique up to a constant term.)

Finally, we can return to our original integral to obtain

$$\int x^2e^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx$$

$$= \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) + C$$

$$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$$

Part 2: Now that we have a formula for $\int x^2e^{2x}dx$, we can compute $\int_1^5 x^2e^{2x}\,dx$ by evaluating the antiderivative at the limits of integration, as follows:

$$\int_1^5 x^2e^{2x}\,dx = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x}\Bigg|_1^5$$

$$= \left(\frac{1}{2}(5)^2e^{2(5)} - \frac{1}{2}(5)e^{2(5)} + \frac{1}{4}e^{2(5)}\right) - \left(\frac{1}{2}(1)^2e^{2(1)} - \frac{1}{2}(1)e^{2(1)} + \frac{1}{4}e^{2(1)}\right)$$

$$= \frac{25}{2}e^{10} - \frac{5}{2}e^{10} + \frac{1}{4}e^{10} - \frac{1}{2}e^2 + \frac{1}{2}e^2 + \frac{1}{4}e^2$$

$$= \frac{41}{4}e^{10} + \frac{1}{4}e^2.$$

Thus, using integration by parts, $\int_1^5 x^2e^{2x}\,dx = \frac{41}{4}e^{10} + \frac{1}{4}e^2$.

2. Evaluate $\int \sin(x) e^x \, dx$.

Solution: For notational purposes, denote

$$I = \int \sin(x) e^x \, dx.$$

In order to compute the integral I, we will use integration by parts via the formula

$$(*) \quad \int f(x) g'(x) \, dx = f(x) g(x) - \int f'(x) g(x) \, dx.$$

To that end, choose $f_1(x) = \sin(x)$ and $g_1'(x) = e^x$. Differentiating $f_1(x)$ gives $f_1'(x) = \cos(x)$ and integrating $g_1'(x)$ gives $g_1(x) = e^x$.

Using the equation $(*)$ above, we obtain

$$(**) \quad I = \int \sin(x) e^x \, dx = \sin(x) e^x - \int \cos(x) e^x \, dx.$$

We will repeat the process to evaluate $\int \cos(x) e^x \, dx$ by choosing $f_2(x) = \cos(x)$ and $g_2'(x) = e^x$. Differentiating $f_2(x)$ gives $f_2'(x) = -\sin(x)$ and integrating $g_2'(x)$ gives $g_2(x) = e^x$.

Using $(*)$, we have

$$\begin{aligned} \int \cos(x) e^x \, dx &= \cos(x) e^x - \int (-\sin(x)) e^x \, dx \\ &= \cos(x) e^x + \int \sin(x) e^x \, dx. \\ &= \cos(x) e^x + I \end{aligned}$$

Noticing that the integral we are looking for has appeared, we return to the equation $(**)$

$$\begin{aligned} I &= \sin(x) e^x - \int \cos(x) e^x \, dx \\ I &= \sin(x) e^x - (\cos(x) e^x + I) \\ I &= \sin(x) e^x - \cos(x) e^x - I. \end{aligned}$$

We can rearrange the equation above to solve for I as follows:

$$I = \sin(x)e^x - \cos(x)e^x - I$$

$$2I = \sin(x)e^x - \cos(x)e^x + C$$

$$I = \frac{1}{2}\left(\sin(x)e^x - \cos(x)e^x\right) + C$$

(We added the "$+C$" at the first stage where we did not have integrals – disguised as I or otherwise – since integrals are only unique up to a constant.)

Therefore, using integration by parts, we have

$$\int \sin(x)e^x\,dx = \frac{1}{2}\left(\sin(x)e^x - \cos(x)e^x\right) + C.$$

Salmon 5

3. Evaluate $\int (\tan(x))^3\,dx$.

To evaluate $\int (\tan(x))^3\,dx$, we will start by replacing $(\tan(x))^2$ according to the Pythagorean identity

$$(\tan(x))^2 = (\sec(x))^2 - 1 .$$

This replacement gives us

$$\begin{aligned}
\int (\tan(x))^3\,dx &= \int \tan(x)\,(\tan(x))^2\,dx \\
&= \int \tan(x)\,((\sec(x))^2 - 1)\,dx \\
&= \int (\tan(x)(\sec(x))^2 - \tan(x))\,dx \\
&= \int \tan(x)(\sec(x))^2\,dx - \int \tan(x)\,dx.
\end{aligned}$$

We can compute $\int \tan(x)(\sec(x))^2\,dx$ by using the substitution

$$u = \tan(x)$$

which forces

$$du = (\sec(x))^2\,dx .$$

Hence,

$$\begin{aligned}
\int \tan(x)(\sec(x))^2\,dx &= \int u\,du \\
&= \frac{u^2}{2} + C \\
&= \frac{(\tan(x))^2}{2} + C .
\end{aligned}$$

The only remaining task is to compute $\int \tan(x)\,dx$. This was done in class by replacing $\tan(x)$ with $\frac{\sin(x)}{\cos(x)}$ and then using substitution. The result is

$$\int \tan(x)\,dx = \ln|\sec(x)| + C .$$

Now, returning to our original integral, we have

$$\int (\tan(x))^3 \, dx = \int \tan(x)(\sec(x))^2 \, dx - \int \tan(x) \, dx$$

$$= \frac{(\tan(x))^2}{2} - \ln|\sec(x)| + C .$$

Salmon 7

4. Evaluate $\int \sqrt{x^2+4x+3}\, dx$.

In order to evaluate $\int \sqrt{x^2+4x+3}\, dx$, we will first rewrite the expression under the square root as a sum or difference of squares. To do so, we will complete the square using the formula

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + c - \left(\frac{b}{2a}\right)^2.$$

Thus, we can write

$$x^2+4x+3 = 1\cdot\left(x+\frac{4}{2\cdot 1}\right)^2 + 3 - \left(\frac{4}{2\cdot 1}\right)^2$$

$$= (x+2)^2 + 3 - (2)^2$$

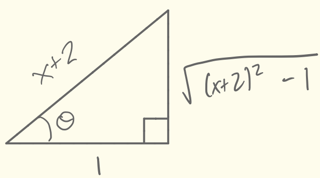
$$= (x+2)^2 + 3 - 4$$

$$= (x+2)^2 - 1.$$

Hence, our integral becomes

$$\int \sqrt{x^2+4x+3}\, dx = \int \sqrt{(x+2)^2-1}\, dx.$$

Now that our integrand is a square root of a difference of squares, we will use a trigonometric substitution. To start, consider the following triangle for reference:



We will use the relation

$$\sec\theta = \frac{x+2}{1}$$

$$\sec\theta = x+2 \quad (\text{or } x = \sec\theta - 2)$$

This forces $dx = \sec\theta\tan\theta\, d\theta$.

Returning to our integral, we can make the substitutions:

$$\int \sqrt{(x+2)^2-1}\, dx = \int \sqrt{(\sec\theta)^2 - 1}\, \sec\theta\tan\theta\, d\theta.$$

Recall the Pythagorian identity

$$(*) \quad (\sec\theta)^2 - 1 = (\tan\theta)^2.$$

Using this identity, we can simplify our integral

$$\int \sqrt{(\sec\theta)^2-1}\, \sec\theta\tan\theta\, d\theta = \int \sqrt{(\tan\theta)^2}\, \sec\theta\tan\theta\, d\theta$$

$$= \int \sec\theta\, (\tan\theta)^2\, d\theta$$

(Since we are computing an indefinite integral, we may assume/assert that $\sqrt{(\tan\theta)^2} = \tan\theta$.)

Now, we must compute the trigonometric integral

$$I = \int \sec\theta (\tan\theta)^2\, d\theta.$$

To that end, we use $(*)$ again to write

$$I = \int \sec\theta\, (\tan\theta)^2\, d\theta = \int \sec\theta\left((\sec\theta)^2 - 1\right) d\theta$$

$$= \int (\sec\theta)^3 - \sec\theta \;\; d\theta.$$

Salmon 8

As computed in class, we know that

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C.$$

All that remains is to compute $\int (\sec(\theta))^3 \, d\theta$. To that end, we will use integration by parts with the formula

$$(**) \quad \int f(x) g'(x) \, dx = f(x)g(x) - \int f'(x) g(x) \, dx$$

with the choice $f_1(\theta) = \sec\theta$ and $g_1'(\theta) = (\sec\theta)^2$. Differentiating $f(x)$, we obtain $f_1'(\theta) = \sec\theta\tan\theta$ and integrating $g_1'(\theta)$ we obtain $g_1(\theta) = \tan\theta$. Using the equation $(**)$, we have

$$\begin{aligned}
\int (\sec\theta)^3 \, d\theta &= \sec\theta\tan\theta - \int (\sec\theta\tan\theta)\tan\theta \, d\theta \\
&= \sec\theta\tan\theta - \int (\tan\theta)^2 \sec\theta \, d\theta . \\
&= \sec\theta\tan\theta - I
\end{aligned}$$

Returning to our last equation involving I, we have

$$\begin{aligned}
I &= \int (\sec\theta)^3 d\theta - \int \sec\theta \, d\theta \\
&= \sec\theta\tan\theta - I - \ln|\sec\theta\tan\theta|
\end{aligned}$$

Rearranging and solving for I, we obtain

$$\begin{aligned}
2I &= \sec\theta\tan\theta - \ln|\sec\theta\tan\theta| + C \\
I &= \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta\tan\theta|\right) + C.
\end{aligned}$$

(We added the "+ C" here since antiderivatives are only unique up to the addition of a constant.)

Therefore, $\int \sec\theta (\tan\theta)^2 d\theta = \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta\tan\theta|\right) + C$.

Returning to our triangle, we need to return to our original variable x. We have

$$\sec\theta = x+2, \qquad \tan\theta = \sqrt{(x+2)^2-1}.$$

So,

$$\begin{aligned}
\int \sqrt{x^2+4x+3}\,dx &= \int \sqrt{(x+2)^2-1}\,dx \\
&= \int \sec\theta\,(\tan\theta)^2\,d\theta \\
&= \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right) + C \\
&= \frac{1}{2}\left((x+2)\sqrt{(x+2)^2-1} - \ln\left|x+2+\sqrt{(x+2)^2-1}\right|\right) + C.
\end{aligned}$$

Salmon 10

5. Evaluate $\displaystyle\int \frac{1}{x^3+2x^2+x}\,dx$

Solution: Noticing that the denominator of the integrand is a polynomial that can be factored, we will start with decomposing the integrand via partial fractions.

Since $x^3+2x^2+x = x(x+1)^2$, we will use the form

$$\frac{1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2}.$$

The denominators are equal so the numerators must be equal also:

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

$$= A(x^2+2x+1) + Bx^2 + Bx + Cx$$

$$= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx$$

$$= (A+B)x^2 + (2A+B+C)x + A$$

The coefficient of x^2 on both sides must be the same so $0 = A+B$. Similarly, the coefficient of x must be the same on both sides so $0 = 2A+B+C$. Lastly, the constant term must be the same on both sides so $1 = A$. The resulting system of equations we must solve is

$$0 = A+B$$
$$0 = 2A+B+C$$
$$1 = A.$$

Solving this system leads to

$$A = 1, \quad B = -1, \quad C = -1.$$

Therefore, our integral becomes

$$\int \frac{1}{x^3+2x^2+x}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx.$$

We we address each of the integrals on the right separately.

First, $\int \frac{1}{x}\,dx = \ln|x| + C.$

Second, choose $u = x+1$. Then $du = dx$ so

$$\int \frac{1}{x+1}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x+1| + C.$$

Third, choose $u = x+1$, again. (So $du = dx$ again also)

Hence, $\int \frac{1}{(x+1)^2}\,dx = \int (x+1)^{-2}\,dx$

$$= \int u^{-2}\,du$$

$$= -u^{-1} + C$$

$$= -(x+1)^{-1} + C$$

$$= \frac{-1}{x+1} + C.$$

Putting this all together, we have

$$\int \frac{1}{x^3+2x^2+x}\,dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C.$$

6. Determine if $\int_{-1}^{1} \frac{1}{x}\,dx$ converges or diverges.

Solution: The improper integral $\int_{-1}^{1} \frac{1}{x}\,dx$ is defined as the sum

$$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

and so $\int_{-1}^{1} \frac{1}{x}\,dx$ will converge only if both $\int_{-1}^{0} \frac{1}{x}\,dx$ and $\int_{0}^{1} \frac{1}{x}\,dx$ converge. To that end, we will start by computing $\int_{0}^{1} \frac{1}{x}\,dx$ using the definition:

$$\begin{aligned}
\int_{0}^{1} \frac{1}{x}\,dx &= \lim_{t\to 0} \int_{t}^{1} \frac{1}{x}\,dx \\
&= \lim_{t\to 0} \left(\ln|x| \Big|_{t}^{1} \right) \\
&= \lim_{t\to 0} \ln|1| - \ln|t| \\
&= -\infty .
\end{aligned}$$

Hence, $\int_{0}^{1} \frac{1}{x}\,dx$ diverges. This fact means $\int_{-1}^{1} \frac{1}{x}\,dx$ diverges as well.

7. Determine if $\int_0^\infty e^{-2x}(\sin(x))^2\,dx$ converges or diverges.

Solution: In order to determine if the given improper integral converges or diverges, we will use the comparison test for integrals. Let $f(x) = e^{-2x}(\sin(x))^2$ and let $g(x) = e^{-2x}$. For x in $[0,\infty)$, $0 \le (\sin(x))^2 \le 1$ and $e^{-2x} \ge 0$, so

$$0 \le e^{-2x}(\sin(x))^2 \le e^{-2x} \quad (\text{that is, } 0 \le f(x) \le g(x)).$$

Further, $f(x)$ and $g(x)$ are continuous for all x in $[0,\infty)$. By the integral comparison test, if $\int_0^\infty e^{-2x}\,dx$ converges, then $\int_0^\infty e^{-2x}(\sin(x))^2\,dx$ converges. We will now investigate the improper integral $\int_0^\infty e^{-2x}\,dx$ using the definition. To that end, consider

$$\begin{aligned}
\int_0^\infty e^{-2x}\,dx &= \lim_{t\to\infty} \int_0^t e^{-2x}\,dx \\
&= \lim_{t\to\infty} \left(\frac{-1}{2}e^{-2x}\Big|_0^t\right) \\
&= \lim_{t\to\infty} \left(\frac{-1}{2}e^{-2t} + \frac{1}{2}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Thus, $\int_0^\infty e^{-2x}\,dx$ converges which gives us that

$\int_0^\infty e^{-2x}(\sin(x))^2\,dx$ also converges.